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Characterizations of Certain Classes of Graphs

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This paper characterizes finite edge regular graphs having the property that any arc can be completed to a triangle such that any vertex not on the triangle is arced to one or three vertices of the triangle. A generalization with an application to finite doubly transitive groups is also given.

INTRODUCTION

Theorem 1 is a characterization of finite edge regular graphs having the property that any arc can be completed to a triangle such that all vertices off the triangle are arced to one or three members of the triangle. The finite graphs which can appear are complete, totally disconnected, the graph of the perpendicular relation on the non-zero vectors in a non-degenerate symplectic geometry over GF(2), or the graph of the perpendicular relation among the singular vectors in one of the two non-degenerate orthogonal geometries over GF(2). Theorem 2, characterizing strongly regular graphs with a weaker condition, allows the above list to be expanded to include the pentagon. Theorem 2 has an application in characterizing certain doubly transitive groups.

All graphs considered here are implicitly assumed to be undirected and without loops. The following theorems characterize certain families of finite regular graphs:

THEOREM 1. *Let $\mathscr{G}$ be a regular graph (undirected, without loops) having a finite number of vertices. Suppose that every arc (a, b) in $\mathscr{G}$ can be completed to a mutually arced triplet (triangle) $\{a, b, c\}$ such that every vertex in $\mathscr{G}$-$\{a, b, c\}$ is arced to 1 or 3 of the vertices in $\{a, b, c\}$. Then $\mathscr{G}$ is isomorphic to one of the graphs, $\mathscr{C}_n$, $\mathscr{D}_n$, $\mathscr{S}_n^{(1)}$, $\mathscr{S}_n^{(2)}$ or $\mathscr{S}\mathscr{P}(2n, 2)$.*

THEOREM 2. *Let $\mathscr{G}$ be a strongly regular graph having a finite number of vertices. For every arc (a, b) and distinguished end vertex a assume there exists a vertex c arced to a such that every vertex not arced to a is arced to exactly one of c or b. Then $\mathscr{G}$ is isomorphic to $\mathscr{C}_n$, $\mathscr{D}_n$, $\mathscr{S}_n^{(1)}$, $\mathscr{S}_n^{(2)}$, $\mathscr{S}\mathscr{P}(2n, 2)$ or is a pentagon.*

THEOREM 3 (Characterization of the pentagon). *Assume $\mathscr{G}$ is a strongly regular finite graph. Assume that, for every arc (a, b) with distinguished vertex a, there exists at least one vertex c such that c is arced to a and every vertex not arced to a is arced to exactly one of b and c. Assume that for at least one arc (a, b) and vertex a, in $\mathscr{G}$ the hypothesized vertex c is not arced to b. Then $|\mathscr{G}| = 5$ and $\mathscr{G}$ is a pentagon.*

In the theorems above, $\mathscr{C}_n$ and $\mathscr{D}_n$ denote the complete and the totally disconnected graphs of n vertices, respectively. The symbol $\mathscr{SP}(2n, 2)$ refers to the graph of $2^{2n} - 1$ vertices formed from the relation of being perpendicular among the non-zero vectors of a non-degenerate symplectic space of dimension $2n$ over GF(2). The symbols $\mathscr{S}_n^{(1)}$ and $\mathscr{S}_n^{(2)}$ denote the graphs of singular vectors under the perpendicular relation in either of the two non-degenerate orthogonal geometries of dimension $2n$ over GF(2).

Under the hypothesis of Theorem 1, $\mathscr{G}$ must be edge regular (Lemma 3.1, ii). Also by Lemma 3.1, iv it follows that either $\mathscr{G}$ is $\mathscr{C}_n$ or $\mathscr{D}_n$ or else $m(B'(x))$ is constant—i.e., for each vertex x in $\mathscr{G}$, the set of vertices distinct from x and not arced to x defines a subgraph $B'(x)$ which is regular with valence v_x not depending on x. In any event the dual graph $\mathscr{G}'$ is also edge regular. Thus $\mathscr{G}$ is strongly regular and so Theorem 2 is a generalization of Theorem 1.

Section 1 contains graph-theoretic terminology and notation used throughout. Section 2 is a brief review of the symplectic and orthogonal geometries over GF(2). Basic terminology (e.g., bilinear form, non-degenerate, etc.) can be found in [1].

Section 3 proceeds under the hypotheses of Theorem 1 (Hypothesis 3.1). Theorem 1 follows readily from the lemmas of this section: Suppose $\mathscr{G}$ satisfies the hypotheses of Theorem 1. If $\mathscr{G} \simeq \mathscr{C}_n$ or $\mathscr{D}_n$ we are done. Otherwise $\mathscr{G}$ contains two vertices a and b which are not arced and for which $B(a) \cap B(b)$ is non-empty by Lemma 3.1, i. If (x, y) is an arc in $B(a) \cap B(b)$, by hypothesis there exists a vertex z such that $\{x, y, z\}$ is a triangle T and every vertex in $\mathscr{G} - T$ is arced to 1 or 3 members of T. Thus $z \in B(a) \cap B(b)$ and so $B(a) \cap B(b)$ is a subgraph of $\mathscr{G}$ which also satisfies Hypothesis 3.1. By Lemma 3.1, ii, $B(a) \cap B(b)$ is regular. Thus, by induction on the cardinality of $B(a) \cap B(b)$, $B(a) \cap B(b) \simeq \mathscr{C}_n, \mathscr{D}_n, \mathscr{S}_n^{(1)}$, $\mathscr{S}_n^{(2)}$ or $\mathscr{SP}(2n, 2)$. By Lemma 1.5, since $\mathscr{G} \not\simeq \mathscr{C}_n$, $B(a) \cap B(b) \not\simeq \mathscr{C}_n$. Lemmas 1.3 and 1.4 show that in the remaining cases $\mathscr{G}$ is of the desired type.

Section 4 operates under Hypothesis 4.1, which is essentially the hypotheses of Theorem 2 except that it does not assume that $\mathscr{G}$ is strongly regular. Then Lemma 4.4 is Theorem 3, and Lemma 4.5, together with Theorem 1, yield Theorem 2.

Section 5 applies Theorem 2 to classify certain doubly transitive groups which occur by virtue of the Graph-Extension Theorem [7].

1. Notation

All graphs considered in this paper contain a finite number of vertices, are undirected, and have no loops. The symbol denoting a graph will also be used to denote the set of its vertices; this convention applies also to subgraphs of a given graph. The arcs of a graph are simply a distinguished subset of the set of unordered pairs of vertices of the graph. For a given vertex x in a graph, the number of distinct arcs of the graph which contain x is called the *valence* of x. A graph $\mathscr{G}$ is called *regular* if all of its vertices have the same valence, and we call this common valence the *valence* of $\mathscr{G}$, denoted $m(\mathscr{G})$. We say "x is arced to y" if (x, y) is an arc. In a graph $\mathscr{G}$, the set of vertices y to which x is arced is denoted $B(x)$ and is called the boundary of x. Throughout, for any set, X, the symbol $|X|$ will denote the cardinality of X. To recapitulate slightly, if $\mathscr{G}$ is regular, then $|B(x)|$ is constant for all $x \in \mathscr{G}$. We write $B'(x)$ for $\mathscr{G} - B(x) - \{x\}$ so that we always have a decomposition $\mathscr{G} = \{x\} + B(x) + B'(x)$. Let $\mathscr{G}$ be a graph and let A and B be subsets (or subgraphs) of $\mathscr{G}$. The symbol $[A, B]$ denotes the set of arcs (a, b) in $\mathscr{G}$ with $a \in A$ and $b \in B$. We write $[a, B]$ for $[\{a\}, B]$, in case A is a singleton. *A path of length m, from vertex a to b is a sequence of vertices* $\{x_0, x_1, x_2, \ldots, x_m\}$ such that $a = x_0$, $x_m = b$ and (x_j, x_{j+1}) are arcs for $j = 0, 1, \ldots, m - 1$. A path of length 0 from a to a is the sequence $\{a\}$. The distance $\rho(a, b)$ is the minimum of all path lengths for paths from a to b. Clearly ρ is a metric on $\mathscr{G}$. We define in the usual way as diam $(\mathscr{G}) = \sup(\rho(a, b) | (a, b) \in \mathscr{G} \times \mathscr{G})$. We write diam $(\mathscr{G}) = \infty$ if $\mathscr{G}$ is disconnected. The statement diam $(\mathscr{G} \geqslant k$ is intended to include the possibility that diam $(\mathscr{G}) = \infty$. If $\mathscr{G}$ is a graph, the dual graph $\mathscr{G}'$ is the graph whose vertices are the vertices of $\mathscr{G}$ and whose arcs are exactly the pairs of distinct vertices which are not arcs of $\mathscr{G}$.

The graph $\mathscr{G}$ is said to be *edge regular* (1) if $B(x)$ is a regular subgraph of $\mathscr{G}$ for every vertex x and (2) if the pair of integers $(|B(x)|, m(B(x)))$ remains constant as x ranges over the vertices of $\mathscr{G}$. (This definition appears in [3].) Clearly an edge regular graph is regular. A graph is said to be *strongly regular* if and only if $\mathscr{G}$ is regular and the integer $|B(x) \cap B'(y)|$, $x \neq y$, depends only on whether (x, y) is an arc or not (see Bose [2]). A finite graph $\mathscr{G}$ is strongly regular if and only if both $\mathscr{G}$ and $\mathscr{G}'$ are edge regular. A graph $\mathscr{G}$ is called *complete* if every unordered pair of distinct vertices of $\mathscr{G}$ is an arc. Clearly any two complete graphs having the same cardinal number of vertices are isomorphic; the symbol

$\mathscr{C}_n$ denote the complete graph with n vertices. A graph is called *totally disconnected* if the set of arcs is empty. Again, such a graph is determined up to isomorphism by the cardinality of its set of vertices; $\mathscr{D}_n$ will denote the totally disconnected graph having n vertices. Certain very small graphs recur frequently in the proofs. A *triangle* is a complete graph with three vertices. A "V" is a graph having three vertices and two arcs. A *diamond* is a regular graph of valence 2 on four vertices. A *pentagon* is a regular graph of valence 2 on five vertices.

2. Geometry over GF(2)

Let V be a vector space over GF(2) which admits a non-degenerate symplectic bilinear form $f: V \times V \to \mathrm{GF}(2)$. Then V is a direct sum of 2-dimensional subspaces P_i, $i = 1,\ldots, n$, such that $f(u, v) = \delta_{ij}$ if $u \in P_i$, $v \in P_j$ and $u \neq v$. The subspaces P_i are called *hyperbolic planes*. There is nothing unique about the decomposition $V = P_1 \oplus \cdots \oplus P_n$. Two vectors u and v are called *perpendicular* if $f(u, v) = 0$. One frequently writes this, $u \perp v$. The symbol $u^{\perp}$ denotes the set of vectors perpendicular to u and is a hyperplane (subspace of codimension 1) in V. The set of non-zero vectors of V form a regular graph under the relation of being perpendicular. This graph is denoted $\mathscr{S}\mathscr{P}(2n, 2)$.

An *orthogonal geometry over* GF(2) is a partition of the non-zero vectors of the non-degenerate symplectic space V of the preceding paragraph into two sets $\mathscr{N}$ and $\mathscr{S}$ called, respectively, the *non-singular* and *singular* vectors of V subject to the following rules:

(1) If $a, b \in \mathscr{S}$, $a \neq b$, and $a \perp b$, then $a + b \in \mathscr{S}$.

(2) If $a, b \in \mathscr{S}$, $a \neq b$ and a is not perpendicular to b, then $a + b \in \mathscr{N}$.

(3) If $a, b \in \mathscr{N}$, $a \neq b$, and $a \perp b$, then $a + b \in \mathscr{S}$.

(4) If $a, b \in \mathscr{N}$, $a \neq b$, and a is not perpendicular to b, then $a + b \in \mathscr{N}$.

(5) If $a \in \mathscr{S}$, $b \in \mathscr{N}$ and $a \perp b$, then $a + b \in \mathscr{N}$.

(6) If $a \in \mathscr{S}$, $b \in \mathscr{N}$ and a is not perpendicular to b, then $a + b \in \mathscr{S}$.

If $\{u_1, \ldots, u_{2n}\}$ is a basis for V, an assignment of the u_i into $\mathscr{S}$ or $\mathscr{N}$ determines uniquely from (1) to (6) an assignment of all the non-zero vectors into $\mathscr{S}$ or $\mathscr{N}$, i.e., determines an orthogonal geometry. There are only two types of assignments for a hyperbolic plane P. Either all three non-zero vectors of P are non-singular or else one is non-singular and two are singular. In the former case P is said to be of *quaternion* type (or type Q). In the latter case P is said to be of *dihedral* type (or type D). In accordance with the first sentence of this paragraph, an orthogonal geometry for V is determined by an assignment of the hyperbolic planes $P_1, \ldots, P_n$ as type Q or D. The orthogonal geometry $O^{(2)}(2n, 2)$ occurs when $V = Q \perp D \perp \cdots \perp D$, (a sum of n mutually perpendicular hyperbolic planes, one of type Q, $n-1$ of type D). The geometry $O^{(1)}(2n, 2)$ occurs when $V = D \perp D \perp \cdots \perp D$ (n mutually perpendicular summands of type D). An appropriate change of basis shows that $D \perp D \simeq Q \perp Q$ and so the above two geometries are essentially the only types possible. The sets $\mathscr{S}$ and $\mathscr{N}$ in $O^{(i)}(2n, 2)$ form graphs under the "perpendicular" relation, $i = 1, 2$. We denote these graphs by the symbols $\mathscr{S}_{n-1}^{(i)}$ and $\mathscr{N}_{n-1}^{(i)}$, respectively, $i = 1, 2$. One may readily calculate that

$$| \mathscr{S}_n^{(1)} | = 2^n(2^{n+1} + 1) - 1 \quad | \mathscr{N}_n^{(1)} | = 2^n(2^{n+1} - 1),$$
$$| \mathscr{S}_n^{(2)} | = 2^n(2^{n+1} - 1) - 1 \quad | \mathscr{N}_n^{(2)} | = 2^n(2^{n+1} + 1).$$

3. Graphs with the Triangle Property

In this section, $\mathscr{G}$ denotes a graph satisfying the following:

HYPOTHESIS 3.1. *$\mathscr{G}$ is a regular graph. If (a, b) is an arc in $\mathscr{G}$, then there exists a third vertex c such that*

(1) *c is arced to both a and b (thus $\{a, b, c\}$ forms a triangle),*

(2) *every vertex in $\mathscr{G} - \{a, b, c\}$ is arced to exactly 1 or 3 members of the triangle $\{a, b, c\}$.*

The complete and totally disconnected graphs on n vertices (the graphs $\mathscr{C}_n$ and $\mathscr{D}_n$, respectively) satisfy the above hypothesis.

LEMMA 3.1. *Assume $\mathscr{G}$ is not a $\mathscr{C}_n$ or a $\mathscr{D}_n$. Then the following statements hold:*

(i) *$\mathscr{G}$ has diameter 2, viz., $\mathscr{G}$ is connected.*

(ii) *$\mathscr{G}$ is strongly regular.*

(iii) *If (a, b) is an arc in $\mathscr{G}$ then there is only one vertex c satisfying conditions (1) and (2) of Hypothesis 3.1 relative to c.*

(iv) *If x and y are distinct vertices which are not arced, then*

$$| B(x) \cap B(y) | = \tfrac{1}{2} | B(x) | .$$

Proof. (i) Since $\mathscr{G}$ is not $\mathscr{D}_n$ and is regular there exists at least one arc leaving each vertex. Given vertex a and arc (a, b) there exists a vertex c

satisfying conditions (1) and (2) of Hypothesis 3.1 relative to (a, b). By condition (2) every vertex different from a has distance 1 or 2 from a. Thus $\mathcal{G}$ has diameter at most 2. Since $\mathcal{G}$ is not $\mathcal{C}_n$ it is exactly 2.

(ii) Fix a vertex a in $\mathcal{G}$, and set $\Gamma = B(a)$. For each vertex x in Γ, set $m(x) = | B(x) \cap \Gamma |$ and $s(x) = | B(x) \cap \Sigma |$ where Σ represents the set of (and subgraph of) vertices in $\mathcal{G}$ not arced to a. Then, since $\mathcal{G}$ is regular.

$$(3.1) \qquad 1 + m(x) + s(x) = | \Gamma | .$$

By Hypothesis 3.1 there exists at least one vertex x' arced to both a and x such that all vertices in $\mathcal{G} - \{a, x, x'\}$ are arced to 1 or 3 members of $\{a, x, x'\}$. Then every member of Σ is either arced to x or x' but not both. Thus we have a partition

$$(3.2) \qquad \Sigma = (\Sigma \cap B(x)) + (\Sigma \cap B(x'))$$

and so

$$(3.3) \qquad \begin{aligned} 1 + m(x') + s(x') &= | \Gamma | , \\ s(x) + s(x') &= | \Sigma | . \end{aligned}$$

But by condition (2) of Hypothesis 3.1, every point of $\Gamma - \{x, x'\}$ which is arced to x is also arced to x'. Thus $m(x) = m(x')$, and so from (3.1), (3.2), and (3.3), $s(x) = s(x') = \frac{1}{2} | \Sigma |$. Thus $m = m(x) = | \Gamma | - \frac{1}{2} | \Sigma | - 1$ does not depend upon the choice of x within Γ and so (since this represents the valence of x with respect to the subgraph Γ which contains it) Γ is a regular subgraph with multiplicity m. But $| \Sigma | = | \mathcal{G} | - | \Gamma | - 1$, and $| \Gamma | = | B(a) |$ does not depend on the choice of a since $\mathcal{G}$ is regular. Thus $B(a)$ is a regular subgraph of multiplicity m for all vertices a in $\mathcal{G}$, i.e., $\mathcal{G}$ is strongly regular.

(iii) Let (a, b) be an arc in $\mathcal{G}$ and suppose both c and c' satisfy Hypothesis 3.1 relative to the arc (a, b). Set $\Gamma = B(a)$, and $\Sigma = \mathcal{G} - (\Gamma \cup \{a\})$ as in the proof of (ii). Then from Hypothesis 3.1 two equations result:

$$(3.4) \qquad \Sigma - (B(b) \cap \Sigma) = B(c) \cap \Sigma = B(c') \cap \Sigma;$$

$$(3.5) \qquad \begin{aligned} \Gamma \cap B(b) - \{b, c\} &= \Gamma \cap B(c) - \{b, c\}, \\ \Gamma \cap B(b) - \{b, c'\} &= \Gamma \cap B(c') - \{b, c'\}. \end{aligned}$$

It follows from (3.5) that c is arced to c'. Since both c and c' are arced to a, we have

$$(3.6) \qquad B(c) - \{c'\} = B(c') - \{c\}.$$

Thus c' is a point in $B(c)$ arced to every other point in $B(c)$. By (ii) $B(c)$ is a regular subgraph. It follows that $B(c)$ is complete. Then since c is arced to vertex a and the $|\Gamma| - 1$ vertices of $B(c) - \{c\}$, c does not belong to any further arcs terminating in $\Sigma = B'(a)$. Moreover, this last statement holds for all c in Γ. It follows that since $\mathscr{G}$ is connected (by (i)), Σ is empty. But then $\mathscr{G} = \{a\} \cup B(a)$ is a complete graph $\mathscr{C}_n$, contrary to the hypothesis of this lemma. Thus the vertex c' cannot exist.

(iv) Suppose y is a vertex not arced to x in $\mathscr{G}$. Then for each vertex w in $B(x)$, there exists a unique vertex w' in $B(x)$ forming a triangle $\{x, w, w'\}$ such that all vertices in $\mathscr{G} - \{x, w, w'\}$ are arced to 1 or 3 members of this triangle. Thus a portion of $B(x)$ into pairs $\{w, w'\}$ is induced and y is arced to exactly one of the members of this pair. It follows that

$$|B(y) \cap B(x)| = \tfrac{1}{2} |B(x)|$$

as desired.

Definition. Given an arc (x, y) in $\mathscr{G}$, by Lemma 3.1, ii, there exists a *unique* vertex c such that every vertex in $\mathscr{G} - \{x, y, c\}$ is arced to one or three members of $\{x, y, c\}$. Since c is uniquely determined by the arc (x, y) we write $c = e(x, y)$. We refer to the triangle $\{x, y, e(x, y)\}$ formed as a *fundamental triangle.* We observe

Lemma 3.2. *Two fundamental triangles in $\mathscr{G}$ have at most one vertex in common. Also the function*

$$e : [\mathscr{G}, \mathscr{G}] \to \mathscr{G}$$

satisfies

(i) $e(x, y) = e(y, x)$ *and*

(ii) $e(x, e(x, y)) = y$.

The following lemma is a consequence of theorems of Seidel [5, page 194; 6, Theorem 14].

Lemma 3.3. *Assume $\mathscr{G}$ is not complete or totally disconnected. Suppose $\mathscr{G}$ contains two vertices a and b which are not arced. Assume*

$$B(a) \cap B(b) \simeq \mathscr{D}_n$$

the totally disconnected graph of n vertices. Then $n = 2$, 3 or 5. In case $n = 2$ or 5, $\mathscr{G}$ is isomorphic to the graph of singular vectors, $\mathscr{S}_2^{(1)}$ or $\mathscr{S}_3^{(2)}$, respectively, the latter corresponding to the 27 vertex graph associated with the 27 lines of a cubic surface. In case $n = 3$, $\mathscr{G} \simeq \mathscr{S}\mathscr{P}(4, 2)$, the graph of 15 non-zero vectors in a 4-space over GF(2) *subject to a non-degenerate symplectic form.*

Proof. Set $\Gamma = B(a)$ and $\Sigma = B'(a)$. Then $b \in \Sigma$. The vertices of Γ are partitioned into pairs $\{v, v'\}$ such that $\{a, v, v'\}$ is a triangle with the property that every vertex in $\mathscr{G}$ outside this triangle is arced to 1 or 3 members of the triangle. In particular, every vertex in Σ is arced to exactly one member of each pair. Thus we may write $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\Gamma_1 = \Gamma \cap B(b)$$

and $\Gamma_2 = \Gamma \cap B'(b)$. Then $\Gamma_1 = B(a) \cap B(b)$ is a totally disconnected graph of n vertices, and Γ contains only the arcs (v, v'). From strong regularity, (Lemma 3.1, ii), for any arc (x, y) in $\mathscr{G}$ we have $B(x) \cap B(y) = \{e(x, y)\}$, a single point. The proof now proceeds by a series of short steps:

(a) *Given $b \in \Sigma$, there exists a bijection*

$$\phi : B(b) \cap \Gamma \to B(b) \cap \Sigma.$$

Set $\phi(v) = e(b, v)$ for all $v \in B(b) \cap \Gamma$, and $\phi^{-1}(w) = e(b, w)$ for all $w \in B(b) \cap \Sigma$. Then since b and a are not arced

$$\phi(B(b) \cap \Gamma) \subseteq B(b) \cap \Sigma,\ \phi^{-1}(B(b) \cap \Sigma) \subseteq B(b) \cap \Gamma,$$

and $\phi\phi^{-1}$ and $\phi^{-1}\phi$ are identity maps on the appropriate domains.

(b) *Every point $z \in B(b) \cap \Sigma$ is arced to $\phi^{-1}(z)$ and is arced to each of the $n - 1$ vertices in*

$$\Gamma - (B(b) \cap \Gamma + e(a, \phi^{-1}(z)).$$

Since $B(b)$ is the graph of $2n$ vertices consisting of n disjoint arcs, each point z in $B(b) \cap \Sigma$ belongs to exactly one arc in $B(b)$, namely, $\phi^{-1}(z)$. It is thus not arced to $e(a, \phi^{-1}(z))$, nor to any further vertex

$$w \in B(b) \cap \Gamma - \{\phi^{-1}(z)\}.$$

It is therefore arced to each of the $n - 1$ vertices of the set

$$\{e(a, w) |\ w \in B(b) \cap \Gamma - \{\phi^{-1}(z)\}\}.$$

This is the set described in (b) and so (b) is proved.

(c) *Let z_1 and z_2 be two vertices in $B(b) \cap \Sigma$. Then z_1 and z_2 are arced to b and are both arced to exactly one further vertex in Σ, i,e.,*

$$|\ B(z_1) \cap B(z_2) \cap \Sigma\ | = 2.$$

Now Σ has the following properties:

(3.7) *Σ contains no triangles.*

(3.8) *Each "v" in Σ can be uniquely completed to a diamond.*

First $B(b) \cap \Sigma$ is a totally disconnected graph of n vertices since for any $z \in B(b) \cap \Sigma$, $(z, \phi^{-1}(z))$ is the unique arc in $B(b)$ containing z, and $\phi^{-1}(z) \notin \Sigma$.

Since z_1 and z_2 are not arced (because of the paragraph) $B(z_1) \cap B(z_2)$ contains n vertices (and no arcs). But from (b)

$$B(z_1) \cap B(z_2) \cap \Gamma = \Gamma - B(b) - \{e(\phi^{-1}(z_1), e(\phi^{-1}(z_2))\},$$

a set of $n - 2$ vertices. Hence $B(z_1) \cap B(z_2) \cap \Sigma$ contains just two vertices, one of which is b. Thus the "v" formed from $\{b, z_1, z_2\}$ is uniquely completeable to a diamond. The rest of the statements in (c) follow upon generalizing from the choice of $\{b, z_1, z_2\}$.

(*d*) *Let $B_2(b)$ denote the set of vertices in Σ which have distance two from b with respect to the subgraph Σ. Then $B_2(b)$ contains $\binom{n}{2}$ vertices each arced to a unique pair of vertices in*

$$B(b) \cap \Sigma.$$

Suppose $w \in B_2(b)$ and suppose z_1, z_2 and z_3 are three vertices in $(B(b) \cap \Sigma)$ which are arced to w. Then $\{w, z_1, b\}$ is a "v" with vertex z_1 which can be completed to two different diamonds $\{w, z_1, b, z_i\}$ $i = 1, 2$. This is contrary to (c) and so there are at most 2 vertices in

$$(B(b) \cap \Sigma) \cap B(w).$$

From the definition of $B_2(b)$ there is at least one vertex z_1 in this intersection and in that case the "v" $\{w, z_1, b\}$ has a unique completion to a diamond $\{w, z_1, b, z_2\}$ and clearly

$$\{z_1, z_2\} = (B(b) \cap \Sigma) \cap B(w).$$

Now from (c) every pair of vertices $\{z_1, z_2\}$ in $B(b) \cap \Sigma$ produces a "v," $\{z_1 \, b \, z_2\}$, with vertex b, which has unique completion to a diamond $\{x_1, a, z_2, w_{12}\}$ with $w_{12} \in B_2(b)$. From the previous paragraph, every w in $B_2(b)$ arises this way. From the uniqueness of the completion of the "v" $\{z_1, a, z_2\}$, we see that there is a 1-1 correspondence between the vertices of $B_2(b)$ and the unordered pairs of vertices in $B(b) \cap \Sigma$.

(e) $n = 2, 3, 4$ or 5. In particular,

$$|\Sigma| \geqslant |\{b\} \cup (B(b) \cap \Sigma) \cup B_2(b)| = 1 + n + n(n-1)/2. \tag{3.9}$$

But from edge regularity of $\mathcal{G}$, and Lemma 3.1, iv, each vertex in Γ belongs to $s = \frac{1}{2} \mid \Sigma \mid$ arcs which terminate at vertices in Σ. Then since a vertex in Γ belongs to only one arc in Γ and is arced to a, we see that

$$2n - 2 = \tfrac{1}{2} \mid \Sigma \mid . \tag{3.10}$$

Then (3.9) and (3.10) yield $n \leqslant 5$. If $n = 1$, $\mathcal{G} \simeq \mathcal{C}_3$, the complete graph of 3 vertices. This is contrary to assumption. Thus $n = 2, 3, 4$, or 5, and we consider these cases separately.

(f) *If $n = 2$, $\mathcal{G}$ is isomorphic to $\mathcal{S}_2^{(1)}$.*

Since $n = 2$, $\mid \Sigma \mid = 4$ and, by (c) (1.8), Σ is a diamond. Since

$$\mid B(x) \cap B(y) \mid = 1 \text{ or } 2$$

according as (x, y) is an arc or non-arc, it follows that members of a common arc in Γ have disjoint boundaries in Σ. Since $B(x)$ is a subgraph consisting of two disjoint arcs, $B(x) \cap \Sigma$ is an arc for each $x \in \Gamma$. This uniquely determines up to isomorphism the way the remaining arcs in $\mathcal{G}$ (namely $[\Gamma, \Sigma]$) must be defined.

(g) *If $n = 3$, $\mathcal{G} \simeq \mathcal{SP}(4, 2)$.*

Here Γ consists of 3 disjoint arcs. For $b \in \Sigma$, $B(b) \cap \Sigma \simeq \mathcal{D}_3$, and $B_2(b)$ consists of 3 points which are arced, respectively, to the 3 pairs which can be chosen among to vertices in $B(b) \cap \Sigma$. By (c)

(1.7) $\quad y_1 \not\sim y_2 \quad \text{if} \quad \{y_1, y_2\} \subseteq B_2(b) \quad \text{and} \quad B(y_1) \cap B(y_2) \cap \Sigma \neq \phi.$

Thus the subgraph $B_2(b)$ contains no arcs. There remains one further vertex in Σ which is not arced to any vertex in $\{b\} \cup (B(b) \cap \Sigma)$ and so is arced to all three members of $B_2(b)$. Thus Σ is determined. It remains to define the arcs of $[\Gamma, \Sigma]$. Let $\Gamma \cap B(b)$ contains vertices labeled 1, 2, 3. Then we may label the vertices of $B(b) \cap \Sigma$ by $\phi(1)$ $\phi(2)$ $\phi(3)$. Then $\phi(x) \sim x$ and $e(a, y) \sim \phi(x)$ if x and y are distinct vertices in $\{1, 2, 3\}$. This defines $[\Gamma, \{b\} \cup (B(b) \cap \Sigma)]$. Now in Σ each vertex v has a unique vertex v' of distance 3 from v relative to the subgraph Σ. Then $B(v) \cap B(v') \cap \Sigma$ is empty, and so $B(v) \cap \Gamma = B(v') \cap \Gamma$. Thus, since every further vertex in Σ has Σ-distance three from some vertex in $\{b\} \cup (B(b) \cap \Sigma)$, all of $[\Gamma, \Sigma)$ is determined.

(h) *If $n = 4$, the graph does not exist.*

In this case b is arced to four vertices in Σ which we label $\{1, 2, 3, 4\}$. There are no arcs among these four vertices. $B_2(b)$ consists of six vertices which may be labeled by unordered pairs (x, y), $x, y \in \{1, 2, 3, 4\}$, and

(x, y) is arced to the vertices labeled x and y in $B(b) \cap \Sigma$. We must have for vertices in $B_2(b)$ that (x, y) is *not* arced to (u, v) if $\{x, y\} \cap \{u, v\}$ is non-empty. Now Σ contains $4(n-1) = 12$ vertices and so one further vertex b' remains. Then b' has distance 3 from b. Σ is a regular subgraph by Lemma 3.1, iv and the regularity of $\mathscr{G}$. Since (x, y) in $B_2(y)$ is arced to x and y and at most one further vertex in $B_2(y)$, it must be that (x, y) is arced to b'. But this means b' is arced to all six members of $B_2(b)$ and this contradicts the fact that Σ is regular with valence n.

(i) *If $n = 5$, $\mathscr{G}$ is the graph of the 27 lines of a cubic surface* (see Coxeter [4]).

Here, Σ contains $16 = 1 + 5 + \binom{5}{2}$ vertices so

$$\Sigma = \{b\} + (B(b) \cap \Sigma) + B_2(b).$$

As before we may label the vertices of $B(b) \cap \Sigma$ with the numbers 1, 2, 3, 4, 5 and represent vertices of $B_2(y)$ as the unordered pairs (x, y), $x, y \in \{1, 2, 3, 4, 5\}$. Then (x, y) is arced to x and y in $B(b) \cap \Sigma$ and must be arced to three other members of $B_2(b)$. These three are represented by the three pairs disjoint from (x, y) in $\{1, 2, 3, 4, 5\}$. Thus Σ is completely determined and the arcs $[\Gamma, \Sigma]$ remain to be determined.

The vertices of Γ are

$$\Gamma \cap B(b) = \{\phi^{-1}(x) \mid x \in B(b) \cap \Sigma\}$$

and

$$\Gamma - (\Gamma \cap B(b)) = \{e(a, \phi^{-1}(x)) \mid x \in B(b) \cap \Sigma\}.$$

For $x \in B(b) \cap \Sigma$, x is arced to $\phi^{-1}(x)$ and every vertex $e(a, \phi^{-1}(y))$ where $y \in B(b) \cap \Sigma - \{x\}$. It remains to determine $[\Gamma, B_2(b)]$. A typical vertex in $B_2(b)$ is labeled (x, y), $x, y \in B(b) \cap \Sigma$. Since $B(b) \cap B(x, y)$ contains no arcs, we see that (x, y) is not arced to $\phi^{-1}(x)$ or $\phi^{-1}(y)$. Yet $B(x, y) \cap B(b)$ contains 5 vertices in $\mathscr{G}$, two (namely x and y) in Σ. Thus $B(x, y) \cap B(b) \cap \Gamma$ has three vertices and is contained in $B(b) \cap \Gamma - \{\phi^{-1}(x), \phi^{-1}(y)\}$. Thus we have (x, y) arced to the three vertices $\phi^{-1}(z)$ where

$$z \in \{1, 2, 3, 4, 5\} - \{x, y\},$$

and to the two vertices $e(a, \phi^{-1}(x))$ and $e(a, \phi^{-1}(y))$. This describes $[\Gamma, B_2(y)]$ and so $\mathscr{G}$ is completely determined.

LEMMA 3.4. *Suppose $\mathscr{G}$ contains two vertices a and b which are not arced. If $B(a) \cap B(b) \simeq \mathscr{S}_n^{(1)}$, $\mathscr{S}_n^{(2)}$ or $\mathscr{S}\mathscr{P}(2n, 2)$, then $\mathscr{G} \simeq \mathscr{S}_{n+1}^{(1)}$, $\mathscr{S}_{n+1}^{(2)}$ or $\mathscr{S}\mathscr{P}(2n+2, 2)$, respectively.*

Proof. Set $\Gamma = B(a)$, $\Sigma = B'(a)$. Then $b \in \Sigma$ and we set $\Gamma_1 = \Gamma \cap B(b)$ and $\Gamma_2 = \Gamma - \Gamma_1$. The mapping

$$\phi : \Gamma_1 \to B(b) \cap \Sigma = \Sigma_1$$

defined by $\phi(x) = e(b, x)$ is a graph isomorphism since, if (x, y) is an arc in Γ_1, then x is arced to at least two members of the fundamental triangle $\{y, b, e(y, b) = \phi(y)\}$ and so $\phi(y)$ is arced to x. Then $\phi(y)$ is arced to both b and x and so is arced to $e(b, x) = \phi(x)$. Thus $(\phi(x), \phi(y))$ is an arc in Σ_1. A similar argument shows that ϕ^{-1} maps $[\Sigma_1, \Sigma_1]$ into $[\Gamma_1, \Gamma_1]$ and so ϕ also preserves non-arcs. Thus ϕ is a graph isomorphism. One further property holds. If $x \in \Gamma_1$ and $y \in \Sigma_1$, then (x, y) is an arc in $\mathscr{G}$ if and only if $(x, \phi^{-1}(y))$ is an arc in Γ. Thus Γ_1 completely determines $[\Gamma_1, \Gamma_1] + [\Gamma_1, \Sigma_1] + [\Sigma_1, \Sigma_1]$.

Now let Σ_2 denote those vertices in Σ of distance 2 from b, distance being measured here relative to the subgraph Σ. Then for z in Σ_2, we have

$$B(z) \cap \Gamma_2 = \Gamma_2 - e(a, B(z) \cap \Gamma_1)$$

and

$$B(z) \cap \Gamma_1 = \Gamma_1 - \phi^{-1}(B(z) \cap \Sigma_1).$$

Thus $B(z) \cap \Gamma$ completely determined, once $B(z) \cap \Sigma_1$ is described.

We now establish:

(a) *If x and y are distinct vertices which are not arced in Σ, then x and y have distance $\geqslant 3$ relative to the subgraph Σ if and only if their boundary in Γ coincide. Indeed,*

$$B(x) \cap \Gamma = B(y) \cap \Gamma \text{ if and only if } B(x) \cap \Sigma \cap B(y) = \phi.$$

First, suppose $x \neq y$, $x, y \in \Sigma$, and $B(x) \cap \Sigma \cap B(y)$ is empty. Then since x is not arced to y

$$| \Gamma \cap B(x)| = | \Gamma \cap B(x) \cap B(y)| = | \Gamma \cap B(y)|,$$

and because of the containment relations connecting these three sets, all are equal, and so $\Gamma \cap B(x) = \Gamma \cap B(y)$.

On the other hand, if $\Gamma \cap B(x) = \Gamma \cap B(y)$, we see that

$$| B(x) \cap B(y)| \leqslant | \Gamma \cap B(x)|$$

and so $B(x) \cap B(y) \subseteq \Gamma$, whence $B(x) \cap B(y) \cap \Sigma$ is empty.

We now tabulate the cardinalities and valencies of the graphs $\mathscr{S}_k^{(1)}$, $\mathscr{S}_k^{(2)}$, $\mathscr{S}\mathscr{P}(2k, 2)$:

(3.11)

$\mathscr{G}$	$\vert \mathscr{G} \vert$	$m(\mathscr{G})$	$\vert \mathscr{G} \vert - m(\mathscr{G}) - 1$
$\mathscr{S}_k^{(1)}$	$2^k(2^{k+1} + 1) - 1$	$2^k(2^k + 1) - 2$	2^{2k}
$\mathscr{S}_k^{(2)}$	$2^k(2^{k+1} - 1) - 1$	$2^k(2^k - 1) - 2$	2^{2k}
$\mathscr{S}\mathscr{P}(2k, 2)$	$2^{2k} - 1$	$2^{2k-1} - 2$	2^{2k-1}

(b) *If $B(a) \cap B(b) \simeq \mathscr{S}_k^{(1)}, \mathscr{S}_k^{(2)}$ or $\mathscr{S}\mathscr{P}(2k, 2)$ then $\vert \Sigma \vert = 2^{2k+2}, 2^{2k+2}$ or 2^{2k+1}, respectively. In each case $m(\Sigma) = \frac{1}{2} \vert \Gamma \vert = \vert B(a) \cap B(b) \vert = \vert \Gamma_1 \vert$ and so*

$$\vert \Sigma \vert \leqslant 3m(\Sigma).$$

For this we observe that Γ is a graph containing $2 \vert B(a) \cap B(b) \vert = 2 \vert \Gamma_1 \vert$ vertices and that

$$m(\Gamma) = 2m(\Gamma_1) + 1.$$

Then each vertex in Γ belongs to s arcs terminating in Σ, with $s = \frac{1}{2} \vert \Sigma \vert$. But $1 + m(\Gamma) + s = \vert \Gamma \vert$ from the regularity of $\mathscr{G}$. Thus

$$\vert \Sigma \vert = 2\,(\vert \Gamma \vert - m(\Gamma) - 1) = 4(\vert \Gamma_1 \vert - m(\Gamma_1) - 1),$$

that is, four times the entry in the last column of the table (3.11). This proves (b).

(c) *Suppose $B(a) \cap B(b) \simeq \mathscr{S}_k^{(1)}, \mathscr{S}_k^{(2)}$ or $\mathscr{S}\mathscr{P}(2k, 2)$. For each vertex y in Σ there exists a at most one vertex y^* in Σ such that y^* is distance 3 from y relative to the subgraph Σ.*

First, suppose y_1 and y_2 are two distinct vertices in Σ such that $B(y_1) \cap \Gamma = B(y_2) \cap \Gamma$. Suppose y_1 is arced to y_2. Then $y_2 \in B(y_1) \cap \Sigma$ and y_2 is arced to every vertex $x \in \Gamma \cap B(y_1)$. Then y_2 is arced to all vertices of $\{e(y_1, x) \vert\, x \in B(y_1) \cap \Gamma\}$, except $y_2 = e(y_1, e(y_1, y_2))$. This last set comprises $B(y_1) \cap \Sigma - \{y_2\}$. Thus y_2 is arced every vertex in $B(y_1) - \{y_2\}$. Since $B(y_1)$ is a regular subgraph of $\mathscr{G}$ by Lemma 3.3(c), we see that $B(y_1)$ is complete. Then so is Γ and so Σ is empty. Then $\mathscr{G} \simeq \mathscr{C}_m$ for some m, contrary to assumption. Thus y_1 is not arced to y_2.

It now follows, using step (a), that, for vertices y_1, y_2 in Σ, $B(y_1) \cap \Gamma = B(y_2) \cap \Gamma$ if and only if $B(y_1) \cap \Sigma \cap B(y_2)$ is empty and y_1 and y_2 are not arced. This last condition is equivalent to asserting that y_1 and y_2 have distance at least three from one another relative to the subgraph Σ. But the previous "if and only if" statement shows that the relation of being distance at least three from one another relative to the subgraph Σ is an equivalence relation on the vertices of Σ.

If we assume $\mathrm{diam}(\Sigma) \geqslant 3$, then for some vertex y, the set of points of distance 3 from y, $\Sigma_3(y)$ is non-empty. If two distinct vertices y_1 and y_2 lie in $\Sigma_3(y)$, then they are also distance at least 3 from each other, since by the previous paragraph being distance at least 3 is an equivalence relation. Then $\{y\} \cup (B(y) \cap \Sigma), \{y_i\} \cup (B(y_i) \cap \Sigma), i = 1, 2$ denote three mutually disjoint subsets of Σ and so

$$|\Sigma| \geqslant 3\,|\{y\} \cup (B(y) \cap \Sigma)| = 3(1 + m(\Sigma)).$$

This contradicts step (b). Thus $\Sigma_3(y)$ consists of a single vertex.

(d) *Suppose* $B(a) \cap B(b) \simeq \mathscr{S}_k^{(1)}, \mathscr{S}_k^{(2)}$ *or* $\mathscr{S}\mathscr{P}(2k, 2)$. *Suppose* $\mathrm{diam}(\Sigma) \geqslant 3$. *Then* $B(a) \cap B(b) \simeq \mathscr{S}\mathscr{P}(2k, 2)$ *for some* k, *and* $\mathscr{G} \simeq \mathscr{S}\mathscr{P}(2k + 2, 2)$.

Since $\mathrm{diam}(\Sigma) \geqslant 3$ we may assume that, for some vertex in Σ, there exists exactly one vertex of distance 3 from it. Without loss of generality we may assume b is this vertex since, for any vertex y in Σ,

$$\Gamma \cap B(y) \simeq \Gamma \cap B(b) = B(a) \cap B(b).$$

We now have $\{b\} \cup (B(b) \cap \Sigma)$ and $\{b^*\} \cup (B(b^*) \cap \Sigma)$ as disjoint subsets of Σ.

If $B(a) \cap B(b) \simeq \mathscr{S}_k^{(1)}$, $|\Sigma| = 2^{2k+2}$, $m(\Sigma) = 2^k(2^{k+1} + 1) - 1$ and $|\Sigma| < 2m(\Sigma) + 2$. Thus, in this case, the decomposition in the last line of the previous paragraph is not possible.

We now suppose $B(a) \cap B(b) \simeq \mathscr{S}_k^{(2)}$ or $\mathscr{S}\mathscr{P}(2k, 2)$. Let

$$U = \Sigma - (\{b\} \cup (B(b) \cap \Sigma) \cup (B(b^*) \cap \Sigma) \cup \{b^*\})$$

so that U represents the vertices of Σ which are of distance at least 2 from *both* b and b^*. Then U has cardinality $|\Sigma| - 2m(\Sigma) - 2$, so $|U| = 2^{k+1}$ if $B(a) \cap B(b) \simeq \mathscr{S}_k^{(2)}$ and U is empty if $B(a) \cap B(b) \simeq \mathscr{S}\mathscr{P}$. Now $B(b) \cap \Gamma = B(b^*) \cap \Gamma = \Gamma_1$. We may then write

$$B(b) \cap \Sigma = \{e(x, b) |\, x \in \Gamma_1\},$$
$$B(b^*) \cap \Sigma = \{e(x, b^*) |\, x \in \Gamma_1\}.$$

Then, by Hypothesis 3.1, $e(x, b^*)$ is arced $e(y, b^*)$, $x, y \in \Gamma_1$, if and only if x is arced to y, if and only if $e(x, b)$ is arced to $e(y, b)$. Now $e(x, b^*)$, a typical element of $B(b^*) \cap \Sigma$, is not arced to b and so, by Hypothesis 1.1 is arced to exactly one of $y \in \Gamma_1$ and $e(b, y) \in B(b) \cap \Sigma$. But it is arced to y only if x is arced to y. Thus we see that $e(x, b^*)$ is arced to all vertices in

$$\{e(y, b) |\, y \in \Gamma_1, x \not\sim y\}$$

and

$$\{e(w, b^*)\, w \in \Gamma_1, w \sim x\}$$

and to b^*. These number $|\Gamma_1|$ vertices and lie in

$$\{b^*\} \cup (B(b^*) \cap \Sigma) \cup (B(b) \cap \Sigma).$$

But $|\Gamma_1| = |B(e(x, b^*)) \cap \Sigma|$ and so we have proved that no vertex in $B(b^*) \cap \Sigma$ is arced to any vertex in U. Similarly no vertex in $B(b) \cap \Sigma$ is arced to a vertex of U. This proves that, for each $u \in U$,

$$B(u) \cap \Sigma \subseteq U.$$

But this is impossible since $|U| < m(\Sigma)$. Thus U is empty,

$$\Sigma = \{b\} \cup (B(b) \cap \Sigma) \cup (B(b^*) \cap \Sigma) \cup \{b^*\}.$$

The arcs $[\Sigma, \Sigma]$ are determined by the rules

(3.12) $$B(e(x, b^*)) \cap B(b) = \{e(y, b) | y \in \Gamma_1 - B(x) - x\},$$

(3.13) $$B(e(x, b^*)) \cap B(b^*) = \{e(y, b) | y \in \Gamma_1 \cap B(x)\},$$

and so

(3.14) $e(x, b)$ is the unique vertex of distance 3 from $e(x, b^*)$ relative to Σ.

The arcs $[\Gamma, \Sigma]$ are completely determined by the rules

(3.15) $$\Gamma_1 = B(b) \cap \Gamma \qquad \Gamma_2 = \Gamma - B(b),$$

(3.16) $$B(e(x, b)) \cap \Gamma_1 = \{x\} \cup (B(x) \cap \Gamma_1),$$

(3.17) $$B(e(x, b)) \cap \Gamma_2 = \{e(a, y) | y \in \Gamma_1 - B(x) - \{x\}\},$$

(3.18) $$B(v) \cap \Gamma = B(v^*) \cap \Gamma,$$

where v^* is the unique vertex in Σ of distance 3 from v relative to Σ.

This completes the proof of step (d) since uniqueness of $\mathscr{G}$ is determined, and $\mathscr{S}\mathscr{P}(2k, 2)$ satisfies all the hypotheses of step (d).

(e) *Assume* $\operatorname{diam}(\Sigma) = 2$ *and set* $\Sigma_1 = B(b) \cap \Sigma$ *and let* Σ_2 *denote the vertices of* Σ *having distance* 2 *from* b, *relative to the subgraph* Σ. *There exists a mapping* $\mu : [\Sigma_2, \Sigma_1] \to \Sigma_1$ *such that, if* $(x, y) \in [\Sigma_2, \Sigma_1]$, *then* $\mu(x, y)$ *is arced to* x *but is not arced to* y, *and every vertex in*

$$B(x) \cap \Sigma_1 - \{y, \mu(x, y)\}$$

is arced to exactly one member of $\{y, \mu(x, y)\}$, *while every vertex* $\Sigma_1 - (B(x) \cap \Sigma_1)$ *is arced to either both or none of* $\{y, \mu(x, y)\}$.

For each arc (x, y) with $x \in \Sigma_2$ and $y \in \Sigma_1$ set $\mu(x, y) = e(b, e(a, e(x, y)))$. Suppose $y_1 \in \Sigma_1$ and y_1 is arced to x. Then, if y_1 is arced to y, it is also arced to $e(x, y)$ and hence is not arced to $e(a, e(x, y))$—since y_1 is not arced to a,—and so is also not arced to $\mu(x, y) = e(b, e(a, e(x, y)))$, since y_1 is arced to b. Similarly, if y_1 is not arced to y, it is arced to $\mu(x, y)$. Now suppose y_1 is not arced to x. Then it is arced to $\mu(x, y)$ according as it is or is not arced to y. Thus μ has the required properties.

(f) *If* diam $(\Sigma) = 2$, $B(a) \cap B(b) \simeq \mathscr{S}_k^{(j)}, j = 1, 2,$ *and* $\mathscr{G} \simeq \mathscr{S}_{k+1}^{(j)}$, $j = 1, 2$, *respectively*.

Suppose, first, that $B(a) \cap B(b) \simeq \mathscr{S}\mathscr{P}(2k, 2)$. Then $|\Sigma| = 2^{2k+1}$, $|\Sigma| = 2^{2k} - 1$ as $\Sigma_1 \simeq \Gamma_1 \simeq \mathscr{S}\mathscr{P}(2k, 2)$ and $|\Sigma_2| = 2^{2k}$. Select $x \in \Sigma_2$. Then, if $B(x) \cap \Sigma_1$ is empty, $B(x) \cap \Sigma \subsetneq \Sigma_2$ and this contradicts diam $\Sigma = 2$. Thus we may find $y \in \Sigma_1 \cap B(x)$ and form $\mu(x, y) \in \Sigma_1$. Then y is not arced to $\mu(x, y)$ and $B(x) \cap \Sigma_1$ consists of those vertices in Σ_1 arced to exactly one of $\{y, \mu(x, y)\}$. But Σ_1 is isomorphic to the graph of non-zero vectors in a non-degenerate symplectic space of dimension $2k$ over GF(2). Thus y and $\mu(x, y)$ correspond to two non-perpendicular vectors generating a hyperbolic plane P and so $B(x) \cap \Sigma_1$ corresponds to

$$(y^{\perp} - P^{\perp}) + (\mu(x, y)^{\perp} - P^{\perp}), \tag{3.19}$$

a set of $(2^{2k-1} - 2^{2k-2}) + (2^{2k-1} - 2^{2k-2}) = 2^{2k-1}$ vertices in Σ_1. Since x was an arbitrary vertex in Σ_2 we have just shown that every vertex in Σ_2 belongs to 2^{2k-1} arcs terminating in Σ_1. On the other hand, since Σ and Σ_1 (by assumption) are both regular subgraphs, every vertex in Σ_1 belongs to a constant number s_1 of arcs terminating in Σ_2. We can count arcs between Σ_1 and Σ_2 two ways to yield

$$|[\Sigma_1, \Sigma_2]| = 2^{2k} \cdot 2^{2k-1} = (2^{2k} - 1) \cdot s_1, \tag{3.20}$$

and this is clearly impossible since s_1 is an integer. Thus $\Sigma_1 \simeq \mathscr{S}\mathscr{P}(2k, 2)$ is not a possible choice.

Since we are assuming in the hypotheses of this lemma that

$$B(a) \cap B(b) \simeq \mathscr{S}\mathscr{P}(2k, 2), \mathscr{S}_k^{(1)} \text{ or } \mathscr{S}_k^{(2)}$$

only the $\mathscr{S}_k^{(j)}$'s now remain. From the remarks just preceding step (a), the graph $\mathscr{G}$ is completely determined once we have some system of labeling the vertices in Σ_2 and have defined the sets of arcs $[\Sigma_1, \Sigma_2]$ and $[\Sigma_2, \Sigma_2]$. Let $\mathscr{N}_k^{(i)}$ denote the graph of non-singular vectors in one of the two non-degenerate orthogonal geometries of dimension $2k$ over GF(2), two vectors being arced if they are perpendicular. If $\Sigma_1 \simeq \mathscr{S}_k^{(i)}$, we shall show that $\Sigma_2 \simeq \mathscr{N}_k^{(i)}$, and that, if $n \in \Sigma_2$, then n is arced to those vertices in Σ_1

which stand for singular vectors which are *not* perpendicular to n. Thus the objectives stated at the beginning of this paragraph will be achieved, and the graph $\mathscr{G}$ will then be completely and uniquely determined.

First we define a mapping $f: \Sigma_2 \to \mathscr{N}_k^{(i)}$. Select $x \in \Sigma_2$. Since diam $(\Sigma) = 2$, $B(x) \cap \Sigma_1$ is non-empty. For each $y \in B(x) \cap \Sigma_1$ there exists a twin $\mu(x, y)$ and, for any $w \in \Sigma_1 - B(x) \cap \Sigma_1$, w is arced to 0 or 2 members of $\{\mu(x, y), y\}$, while if $w \in B(x) \cap \Sigma_1$ either $w = y$, $w = \mu(x, y)$ or w is arced to exactly one of y and $\mu(x, y)$. Regarding the elements of Σ_1 as singular vectors in $\mathscr{S}_k^{(i)}$, we see, since y is not arced to $\mu(x, y)$, that $y + \mu(x, y)$ stands for a particular non-singular vector n in $\mathscr{N}_k^{(i)}$, and we write $n = f(x)$.

Is f well defined? Suppose y_1, $e(x, y_1)$ are elements of $\Sigma_1 \cap B(x)$. Then $\Sigma_1 \cap B(x) - \{y_1, e(x, y_1)\}$ still represents those vectors in $\mathscr{S}_k^{(i)}$ perpendicular to exactly one of y_1, $e(x, y_1)$. Thus $\Sigma_1 \cap B(x)$ represents those singular vectors which are not perpendicular to $n = y + \mu(x, y)$ on the one hand and $n_1 = y_1 + \mu(x, y_1)$ on the other. Then $y + n$ and $y + n_1$ represent two members of $\mathscr{S}_k^{(i)}$ with the same boundary. But, since $\mathscr{S}_k^{(i)}$ satisfies Hypothesis 1.1, it is easy to see that this forces $y + n = y + n_1$ and so $n = n_1$. Thus the non-singular vector n which is produced does not depend on the choice of y in $\Sigma_1 \cap B(x)$, but only in x. Thus we write $n = f(x)$ and f is well-defined.

Also, from its definition, $f: \Sigma_2 \to \mathscr{N}_k^{(i)}$ has the property that $B(x) \cap \Sigma_1$ are those vectors of $\mathscr{S}_k^{(i)}$ which are *not* perpendicular to $f(x)$.

Suppose, for two vertices x_1, x_2 in Σ_2, $f(x_1) = f(x_2)$. Then $B(x_1) \cap \Sigma_1 = B(x_2) \cap \Sigma_1$. Then $B(x_1) \cap \Gamma = B(x_2) \cap \Gamma$. If they are not arced, $B(x_1) \cap \Sigma \cap B(x_2) = \phi$. But this would not be possible since this intersection is just $\Sigma \cap B(x_1)$ which contains $\Sigma_1 \cap B(x_1)$ and these represent the set of singular vectors not perpendicular to the non-singular vector $f(x_1)$ and $\mathscr{S}_k$ is non-empty.

Since $|\Sigma_2| = |\mathscr{N}_k^{(i)}|$ if $|\Sigma_1| = |\mathscr{S}_k^{(i)}|$, f is onto, and so is a bijection. We may thus use f to "label" the vertices of Σ_2 as non-singular vectors and $[\Sigma_1, \Sigma_2]$ is completely determined by the rule, (x, y) is an arc in $[\Sigma_1, \Sigma_2]$ if and only if $x \not\perp f(y)$. It remains to show that $[\Sigma_2, \Sigma_2]$ is determined.

We need at this point to establish an elementary fact concerning orthogonal geometries over GF(2):

(3.21) *Let n_1 and n_2 be two non-singular vectors in a non-degenerate orthogonal space over* GF(2). *Then there exists a singular vector s with the property that s is perpendicular precisely to those singular vectors perpendicular to none or both of n_1 and n_2 if and only if n_1 and n_2 are mutually perpendicular.* In other words, whether n_1 and n_2 are *perpendicular can be decided by inspecting* $n_1^{\perp} \cap \mathscr{S}$ and $n_2^{\perp} \cap \mathscr{S}$.

To establish (3.21) first suppose n_1 and n_2 are two distinct non-singular vectors which are perpendicular. Then $s = n_1 + n_2$ has all the desired qualities. On the other hand, suppose s is a singular vector with the property s is perpendicular to all singular vectors perpendicular to both or neither of a pair of non-singular vectors $\{n_1, n_2\}$, and is not perpendicular to those singular vectors perpendicular to only one of them. Since s is perpendicular to itself, by hypothesis s is perpendicular to both or neither of $\{n_1, n_2\}$; in any case, s is perpendicular to $n_1 + n_2$. In addition, s is perpendicular to every vector v perpendicular to one or both of n_1 and n_2—that is to say, every vector v perpendicular to $n_1 + n_2$. Thus

$$s \in \operatorname{rad}((n_1 + n_2)^{\perp}) = \langle n_1 + n_2 \rangle$$

and so $s = n_1 + n_2$ is singular. Thus n_1 is perpendicular to n_2 as required.

Now, if (x_1, x_2) is an arc in Σ_2, we may form $e(x_1, x_2)$. Since a and b are each not arced to either x_1 or x_2, $e(x_1, x_2)$ lies in $B(a) \cap B(b) = \Gamma_1$, and we may then form $\phi(e(x_1, x_2)) = e(b, e(x_1, x_2))$ in $\Sigma_1 \simeq \mathscr{S}_k^{(i)}$. Then, since x_1 and x_2 are both arced to $e(x_1, x_2)$ but not b, neither is arced to $e(x_1, x_2)$. Indeed, if $y \in \Sigma_1$ is arced to neither or both of $\{x_1, x_2\}$, then y *is* arced to $e(x_1, x_2)$ as well as b, and so is arced to $e(b, e(x_1, x_2)$ or is equal to it. Recalling from above that (x_j, y) is an arc in $[\Sigma_2, \Sigma_1]$ if and only if the non-singular vector $f(x_j)$ is not perpendicular to the singular vector corresponding to y in the isomorphism $\Sigma_1 \simeq \mathscr{S}_k^{(i)}$, we see that the singular vector corresponding to $\phi(e(x_1, x_2))$ in $\Sigma_1 \simeq \mathscr{S}_k^{(i)}$ has the property of being perpendicular to precisely those singular vectors which perpendicular to one or both of $f(x_1)$ and $f(x_2)$. By (3.21), $f(x_1)$ is perpendicular to $f(x_2)$ in $\mathscr{N}_k^{(i)}$.

Since Σ is regular, and for each $x \in \Sigma_2$, $B(x) \cap \Sigma_1$ is determined by $\mathscr{S}_k^{(i)} - (f(x)^{\perp} \cap \mathscr{S}_k^{(i)})$, the multiplicity of the subgraph Σ_2 is determined by

$$m(\Sigma_2) = \tfrac{1}{2} \mid \Gamma \mid - \rho_k^{(i)} = \mid \mathscr{S}_k^{(i)} \mid - \rho_k^{(i)}, \tag{3.22}$$

where $\rho_k^{(i)}$ denotes the number of singular vectors in $\mathscr{S}_k^{(i)}$ not perpendicular to a given non-singular vector in $\mathscr{N}_k^{(i)}$. It is easy to check that this is also $m(\mathscr{N}_k^{(i)})$. Thus we see that Σ_2 and $\mathscr{N}_k^{(i)}$ have the same multiplicity as regular graphs. Since (x_1, x_2) is an arc in Σ_2 only if $f(x_1)$ is perpendicular to $f(x_2)$, by the previous sentence, "if and only if" also holds. Thus f is a graph isomorphism and $[\Sigma_2, \Sigma_2]$ is uniquely determined.

It now follows that the entire graph $\mathscr{G}$ is uniquely determined up to isomorphism if $B(a) \cap B(b) \simeq \mathscr{S}_k^{(1)}, \mathscr{S}_k^{(2)}$ or $\mathscr{S}\mathscr{P}(2k, 2)$ and $\operatorname{diam}(B'(a)) = 2$. The proof of step (f) is now complete.

(g) $\operatorname{diam}(\Sigma) \geqslant 2$.

If $\operatorname{diam}(\Sigma) = 1$, Σ is complete and so $\Sigma_1 = B(b) \cap \Sigma = \Sigma - \{b\}$ is complete. Then $\Gamma_1 = \phi^{-1}(\Sigma_1)$ is also complete. But this is impossible if $\Gamma_1 = B(a) \cap B(b) \simeq \mathscr{S}_k^{(i)}$ of $\mathscr{S}\mathscr{P}(2k, 2)$ unless $\mathscr{S}_k^{(i)} = \mathscr{S}_1{}^1$ which consists of a single vertex. In this case Σ is empty, and $\mathscr{G}$ is a triangle. But this contradicts our assumption that $\mathscr{G}$ contains two vertices which are not arced.

The conclusion of Lemma 3.4 now follows immediately from steps (g), (f), and (e).

4. A Generalization

Throughout this section the graph $\mathscr{G}$ satisfies the following:

HYPOTHESIS 4.1. *$\mathscr{G}$ is an edge regular graph. Given any arc (a, b) in $\mathscr{G}$, with distinguished vertex a, there exists at least one further vertex c in $\mathscr{G} - \{a, b\}$, arced to a, having the property that any vertex not arced to a is arced to exactly one of the vertices b and c.*

In this hypothesis, the question whether c is arced to b is left moot. As shall be seen in this section, if the dual of $\mathscr{G}$ is also strongly regular the possibilities are quite limited.

We refer to an arc with distinguished end-vertex as a "flag." (This is consistent with the traditional usage of the term "flag" if we view the undirected graph $\mathscr{G}$ as a tactical configuration with block size $k = 2$.) We denote the flag whose arc is (a, b) and distinguished vertex is a by the expression $(a, (a, b))$. According to Hypothesis 2.1, whenever we encounter a flag $(a, (a, b))$ in $\mathscr{G}$, there exists a vertex c such that (a, c) is an arc, and $B'(a)$ is partitioned into two disjoint sets, $B'(a) \cap B(b)$ and $B'(a) \cap B(c)$. For each flag $(a, (a, b))$ it will be convenient to let the symbol $a \circ b$ denote the vertex c with these hypothesized properties. (Note that the distinguished vertex is always written first in $a \circ b$.) Whenever $(a, (a, b))$ is a flag, an element $a \circ b$ with the required properties always exists, but it is not always unique. Since the graph is finite a choice of such a vertex $a \circ b$ can be made at each flag $(a, (a, b))$.

We begin with

LEMMA 4.1. *Set $|\Sigma| = |B'(a)|$. Since $\mathscr{G}$ is regular, this number is constant for all $a \in \mathscr{G}$. Given an arc (a, b), $|[b, B'(a)]| = \frac{1}{2}|\Sigma|$; that is, b belongs to $\frac{1}{2}|\Sigma|$ arcs whose other vertex is not arced to a.*

Proof. For each vertex $v \in B(a)$, set $m(v) = |[v, B(a)]|$ and

$$s(v) = |[v, B'(a)]|.$$

Since $B(v)$ is partitioned as $\{a\} + B(v) \cap B(a) + B(v) \cap B'(a)$, we have

$$(4.1) \qquad 1 + m(v) + s(v) = |\,\Gamma\,|$$

from the regularity of $\mathscr{G}$. But from our definition of $a \circ b$ we see that

$$(4.2) \qquad s(a \circ b) + s(b) = |\,B'(a)| = |\,\Sigma\,|.$$

Edge regularity of $\mathscr{G}$ implies $m(a \circ b) = m(b)$. Then (4.1) implies $s(a \circ b) = s(b)$. Then, from (4.2), $s(b) = \frac{1}{2}\,|\,\Sigma\,|$ as was to be shown.

LEMMA 4.2. *From the flag $(a, (a, b))$ in $\mathscr{G}$ form $a \circ b$ and suppose $a \circ b$ is not arced to b. Form $b \circ a$ from the flag $(b, (a, b))$. Then $b \circ a$ is not arced to a.*

Proof. Suppose, by way of contradiction, that $b \circ a$ is arced to a. Since $a \circ b$ is not arced to b (by hypothesis of this lemma), it is not equal to $b \circ a$, and is arced to exactly one of a and $b \circ a$. Since it is already arced to a, it is not arced to $b \circ a$. Thus we have the subgraph illustrated in Fig. 1.

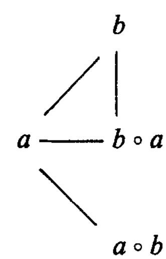

FIGURE 1

Set $\Gamma = B(a)$ and $\Sigma = B'(a)$. Then Σ is partitioned as

$$B(b) \cap \Sigma + B(a \circ b) \cap \Sigma,$$

each set having cardinality $\frac{1}{2}\,|\,\Sigma\,|$, by Lemma 4.1. Since each vertex in $B(a \circ b) \cap \Sigma$ is not arced to either b or a, it is arced to $b \circ a$, whence $B(a \circ b) \cap \Sigma \subseteq B(b \circ a) \cap \Sigma$. Applying edge regularity once more, the cardinality of these two sets must be equal, so

$$(4.3) \qquad B(a \circ b) \cap \Sigma = B(b \circ a) \cap \Sigma.$$

Since $b \not\sim a \circ b$, $\mathscr{G}$ is not complete, and so $\frac{1}{2}\,|\,\Sigma\,| > 1$ by regularity of $\mathscr{G}$. Thus we may select $x \in B(b) \cap \Sigma$ and form $x \circ b$. Now $a \sim b$, $a \not\sim x$. Thus $a \not\sim x \circ b$, so $x \circ b \in \Sigma$. Also $a \circ b \not\sim b$, $a \circ b \not\sim x$, so $a \circ b \sim x \circ b$. Thus $x \circ b \in B(a \circ b) \cap \Sigma$. Then $x \circ b \sim b \circ a$ by (4.3). But now $b \circ a$ is not arced x, also by (4.1) and our partition on Σ, yet $b \circ a$ is arced to both b and $x \circ b$. This contradicts the definition of $x \circ b$ and completes the proof.

LEMMA 4.3. *Assume that, for the flag $(a, (a, b))$, the vertex $a \circ b$ is not arced to b. Then $\mathscr{G}$ has diameter 2 and the dual graph $\mathscr{G}'$ is defined on the same set of vertices as $\mathscr{G}$ with the arc-relation in $\mathscr{G}'$ being that of being distance 2 in $\mathscr{G}$. If $\mathscr{G}'$ is also edge regular, then $\mathscr{G}$ has the following properties:*

(i) *$B'(x)$ is a complete subgraph of $\mathscr{G}$ for each vertex x in $\mathscr{G}$.*

(ii) *Set $|\Gamma| = |B(x)|$, $|\Sigma| = |B'(x)|$ for all x in $\mathscr{G}$, so that $1 + |\Gamma| + |\Sigma| = |\mathscr{G}|$. Then $|\Sigma| = \frac{1}{2}|\Gamma| + 1$.*

(iii) *Setting $S = \frac{1}{2}|\Sigma| - 1$, we have $|\Gamma| = 4S + 2$, $|B(x) \cap B(y)| = 3S$ if x and y are arced, and $|B(x) \cap B(y)| = 2S + 1$ if x and y are not arced.*

Proof. From the existence of the flag $(a, (a, b))$, and the regularity of $\mathscr{G}$, each vertex x in $\mathscr{G}$ is a distinguished vertex of some flag and so by Hypothesis 4.1, every further vertex is either arced to x or to one of two vertices in $B(x)$. Thus $\mathscr{G}$ has diameter $\leqslant 2$. Since $a \circ b$ is not arced to b, diameter of $\mathscr{G}$ is at least 2.

We set $\Gamma = B(a)$ and $\Sigma = B'(a)$ for the remainder of this proof (this notation is consistent with (ii) and (iii) of the statement of the theorem). Edge regularity of $\mathscr{G}$ implies $|B(x) \cap B(y)| = m$ is constant for all arcs (x, y). Every vertex of Γ is arced to $\frac{1}{2}|\Sigma|$ vertices of Σ by Lemma 2.1 and so $m = |\Gamma| - \frac{1}{2}|\Sigma| - 1$. Edge regularity of $\mathscr{G}'$ implies that, for each vertex x, each vertex in $B'(x)$ is arced a constant number of further vertices in $B'(x)$ and hence this number subtracted from $|\Gamma| = |B(x)|$ represents r_x, the number of arcs leaving each vertex in $B'(x)$ for vertices in $B(x)$. Thus $|\Gamma|(\frac{1}{2}|\Sigma|) = r_x \cdot |B'(x)|$ so $r_x = \frac{1}{2}|\Gamma|$ does not depend on x. Thus in general $|B(x) \cap B(y)| = \frac{1}{2}|\Gamma|$ if x is not arced to y.

At this stage (ii) implies (iii), so we only need to prove (i) and (ii).

From Lemma 2.2, since $a \circ b$ is not arced to b, we have a is not arced to $b \circ a$ so $b \circ a \in B(b) \cap \Sigma$ and Σ partitions into two halves, $B(a \circ b) \cap \Sigma$ and $B(b) \cap \Sigma$ of $\frac{1}{2}|\Sigma|$ points each. For each $x \in B(a \circ b) \cap \Sigma$, x is not arced to a or b and hence is arced to $b \circ a$. Thus

$$B(a \circ b) \cap B(b \circ a) \cap \Sigma = B(a \circ b) \cap \Sigma \tag{4.4}$$

and consists of $\frac{1}{2}|\Sigma|$ vertices.

Set $f = |B(b \circ a) \cap B(b) \cap \Gamma|$. By Hypothesis 4.1,

$$B(b \circ a) \cap \Gamma \subseteq B(b). \tag{4.5}$$

Since $m = |B(b) \cap \Gamma| = |B(b) \cap B(b \circ a)|$ and we obtain

$$(4.6) \qquad |B(b) \cap \Gamma - B(b \circ a)| = |B(b) \cap \Sigma \cap B(b \circ a)| = m - f.$$

Now we have a decomposition

$$(4.7) \qquad B(a \circ b) \cap B(b \circ a) \\ = B(a \circ b) \cap B(b \circ a) \cap \Sigma + B(a \circ b) \cap B(b \circ a) \cap \Gamma,$$

the first summand on the right containing $\frac{1}{2}|\Sigma|$ vertices by (4.4), the second containing at most f vertices. The left side contains $\frac{1}{2}|\Gamma|$ vertices since $a \circ b$ and $b \circ a$ are not arced. From this and (4.5):

$$(4.8) \qquad \tfrac{1}{2}|\Gamma| - \tfrac{1}{2}|\Sigma| \leqslant f.$$

Now $\{a\} \cup (B(b) \cap B(a \circ b) \cap \Gamma) = B(b) \cap B(a \circ b)$. Thus $B(b) \cap B(a \circ b) \cap \Gamma$ contains $\frac{1}{2}|\Gamma| - 1$ vertices; of these $\frac{1}{2}|\Gamma| - \frac{1}{2}|\Sigma|$ are also in $B(b \circ a)$ by (4.7) and (4.5), the remaining among the $m - f$ vertices of $B(b) \cap \Gamma - B(b \circ a)$. Thus

$$(4.9) \qquad \tfrac{1}{2}|\Gamma| - 1 \leqslant (\tfrac{1}{2}|\Gamma| - \tfrac{1}{2}|\Sigma|) + (m - f).$$

Thus

$$(4.10) \qquad \tfrac{1}{2}|\Sigma| - 1 \leqslant m - f \leqslant \tfrac{1}{2}|\Sigma| - 1,$$

where the second inequality emerges from (4.4) and the second part of (4.6). Thus $m - f = \frac{1}{2}|\Sigma| - 1$ and so $B(b \circ a) \cap \Sigma = \Sigma - \{(b \circ a)\}$. It follows that Σ is a complete subgraph since it is regular and contains a vertex arced to all other vertices of Σ. Since strong regularity of $\mathscr{G}'$ implies all $B'(x)$ have the same cardinality and multiplicity, (i) holds.

Since each vertex in Σ is arced to $\frac{1}{2}|\Gamma|$ other vertices in Σ, from completeness of Σ, $|\Sigma| - 1 = \frac{1}{2}|\Gamma|$ and (ii) holds. The proof is complete.

LEMMA 4.4. *Suppose for some flag $(a, (a, b))$ and choice of vertex $a \circ b$ that $a \circ b$ is not arced to b. If $\mathscr{G}$ is strongly regular, then $\mathscr{G}$ is a pentagon.*

Proof. All of the conclusion of Lemma 4.3 holds for $\mathscr{G}$. Set $\Gamma = B(a)$ and $\Sigma = B'(a)$. Writing $S = \frac{1}{2}|\Sigma| - 1$, we have $|B(x) \cap B(y)| = 2S + 1$ if $x \not\sim y$ ($= 3S$ if $x \times y$).

Consider a vertex z in Γ and suppose x and y are two distinct vertices in $B'(z) \cap \Gamma$. Then $|B(z) \cap B(x)| = \frac{1}{2}|\Gamma|$. But $B(z) \cap \Gamma$ and $B(x) \cap \Gamma$ are subsets of Γ of size $3S$. But, as z is not arced to x, $B(z) \cap B(x) \cap \Gamma$ contains at least $2S$ elements. But

$$B(z) \cap B(x) = (B(z) \cap B(x) \cap \Gamma) + (B(z) \cap B(x) \cap \Sigma) + \{a\}$$

is a decomposition of a set of $2S + 1$ elements. It follows that

$$B(z) \cap B(x) \cap \Sigma = \phi$$

and so $B(x) \cap \Sigma \subseteq B'(z) \cap \Sigma$. These last sets are equal since they both contain $S + 1$ vertices. A similar argument holds for z and y. Thus

$$B(x) \cap \Sigma = B'(z) \cap \Sigma = B(y) \cap \Sigma,$$

each containing $S + 1$ vertices. Then

$$| B(x) \cap B(y)| \cap \Gamma = | B(x) \cap B(y)| - S - 2 \leqslant 2S - 2.$$

But, if $x \not\sim y$, $| B(x) \cap B(y) \cap \Gamma | \geqslant 2S$. Thus $x \sim y$, and $| B(x) \cap B(y)| = 3S$. Of these $3S$ vertices, $S + 1$ lie in Σ, a is one of them, and so $3S - (S + 2) = 2S - 2$ of them lie in Γ. Since $| B(x) \cap \Gamma | = | B(y) \cap \Gamma | = 3S$, each of the sets $B(x) \cap \Gamma - B(y)$ and $B(y) \cap \Gamma - B(x)$ contain $S + 2$ vertices. But this forces

$$\begin{aligned}&(B(x) \cap \Gamma) \cup (B(y) \cap \Gamma)\\ &\qquad = (B(x) \cap B(y) \cap \Gamma) + (B(x) \cap \Gamma - B(y)) + (B(y) \cap \Gamma - B(x))\end{aligned}$$

to consist of $(2S - 2) + (S + 2) + (S + 2) = 4S + 2 = | \Gamma |$ points. Hence

$$(B(x) \cap \Gamma) \cup (B(y) \cap \Gamma) = \Gamma.$$

But this asserts that every vertex of Γ is either arced to x or arced to y. But this is a contradiction, since z lies in Γ and is not arced to either x or y.

As a consequence of this contradiction, the supposition in the first sentence of the preceding paragraph cannot be maintained. Thus $| B'(z) \cap \Gamma | \leqslant 1$. But $| B'(z) \cap \Gamma | = (4S + 2) - 3S - 1 = S + 1$. Thus $S = 0 = \frac{1}{2} | \Sigma | - 1$. Thus $| \Sigma | = 2$, whence $| \Gamma | = 2$, and $\mathscr{G}$ is a pentagon.

LEMMA 4.5. *Suppose $\mathscr{G}$ is strongly regular and that $\mathscr{G}$ is not a pentagon. Then $\mathscr{G}$ satisfies Hypothesis* 3.1.

Proof. By Lemma 4.4, whenever $(a, (a, b))$ is a flag in $\mathscr{G}$, $a \circ b$ is arced to b. Thus $\{a, b, a \circ b\}$ forms a triangle. Edge regularity of $\mathscr{G}$ implies $| B(x) \cap B(y)|$ is a constant m whenever (x, y) is an arc. Hypothesis 4.1 asserts that $B'(a) \cap B(b) \cap B(a \circ b)$ is empty. Thus $B(b) \cap B(a \circ b) \subseteq \{a\} \cup B(a)$. But $| B(b) \cap B(a \circ b)| = m = | B(a) \cap B(b)|$. It follows that every vertex in Γ which is arced to b either is $a \circ b$ or is a vertex arced to $a \circ b$. Thus $|\{a, b, a \circ b\} \cap B(x)| = 1$ or 3 for all $x \in \mathscr{G}$. Thus the (a, b) lies in a triangle $\{a, b, a \circ b\}$ satisfying the conditions required by Hypothesis 3.1.

5. An Application

The following theorem appears in [7].

Graph Extension Theorem. *Let $\mathscr{G}$ be an undirected graph. Assume*

(1) $G = \mathrm{Aut}(\mathscr{G})$ *is transitive on the vertices of* $\mathscr{G}$.

(2) *For a decomposition* $\mathscr{G} = \{x\} + B(x) + B'(x)$ *there exist mappings*:

$$h_1 : B(x) \to B(x),$$
$$h_2 : B'(x) \to B'(x),$$

such that

(a) *the* h_i *are graph automorphisms of the subgraphs* $B(x)$ *and* $B'(x)$, *respectively*;

(b) *if* $u \in B(x)$, $v \in B'(x)$, *then* (u, v) *is an arc if and only if* $(h_1(u), h_2(v))$ *is not an arc.*

Then there exists a doubly transitive group G^* *such that the subgroup fixing a letter* α, *viewed as a permutation group on the remaining letters, is permutation isomorphic to* G *acting on the vertices of the graph* $\mathscr{G}$.

Our result is the following:

Theorem 4. *In the Graph Extension Theorem assume that* $\mathscr{G}$ *is finite and that either* h_1 *or* h_2 *can also be induced on* $B(x)$ *or* $B'(x)$ *by an automorphism of* $\mathscr{G}$ *fixing* x. *Then* $\mathscr{G}^*$ *is one of the following permutation groups*:

(1) *The symmetric group on n letters.*

(2) *The group* $\mathscr{SP}(2n, 2)$ *in either one of its two doubly transitive representations over the cosets of one of its orthogonal subgroups.*

(3) *The semidirect product of* $\mathscr{SP}(2n, 2)$ *and the additive group of its vector space, viewed as linear substitutions of the vectors in the vector space.*

(4) PSL(2, 5) *acting on six letters.*

Proof. Suppose $\mathscr{G} \simeq \mathscr{C}_{n-1}$ or $\mathscr{D}_{n-1}$. Then $G = \mathrm{Aut}\,\mathscr{G}$ is the symmetric group on $n - 1$ letters and the transitive extension G^* is forced to be the symmetric group on n letters. Thus without loss of generality we may assume that $\mathscr{G}$ is not $\mathscr{C}_n$ or $\mathscr{D}_n$.

Put $x = b$ and write $\Gamma = B(x)$, $\Sigma = B'(x)$. We may assume both Γ and Σ non-empty by the previous paragraph. If g is any automorphism of $\mathscr{G}$ leaving b fixed, then g leaves both Γ and Σ invariant and may be composed with h_1 and h_2 to yield maps h_1', h_2' also satisfying the hypotheses (2) (a) and (2) (b) of the Graph Extension Theorem. Thus by the hypotheses of this theorem we may assume that one of h_1 or h_2 is the identity mapping on its domain. Since h_1 and h_2 also satisfy Hypotheses (2) (a) and (2) (b) relative to the dual graph $\mathscr{G}'$, without loss of generality we may assume $h_2 = 1_\Sigma$. Now consider an arc (b, y). Then $h_1(y)$ is arced to b and any vertex v not arced to b lies in Σ and so is fixed by h_2. Then $h_2(v) = v$ is arced to exactly one of $\{y, h_1(y)\}$. Since G is transitive on $\mathscr{G}$, $\mathscr{G}$ is regular. But for each $v \in \Sigma$, $h_1 : B(v) \cap \Gamma \to \Gamma - (B(v) \cap \Gamma)$ is epimorphic and since h_1 is an automorphism of Γ, $| B(v) \cap \Gamma | = \frac{1}{2} | \Gamma |$. Since $| B(v)| = | \Gamma |$ from the regularity of $\mathscr{G}$, $| B(v) \cap \Sigma | = \frac{1}{2} | \Gamma |$ as so Σ is regular with multiplicity $\frac{1}{2} | \Gamma |$. This asserts that the dual graph $\mathscr{G}'$ is edge regular. For $y \in \Gamma$ write $m(y) = | B(y) \cap \Gamma |$ and $s(y) = | B(y) \cap \Sigma |$. From regularity of $\mathscr{G}$,

$$1 + m(y) + s(y) = | \Gamma | . \tag{5.1}$$

But

$$(B(h_1(y)) \cap \Sigma) + (B(y) \cap \Sigma) = \Sigma. \tag{5.2}$$

Thus

$$s(y) + s(h_1(y)) = | \Sigma | . \tag{5.3}$$

But since $h_1 \in \mathrm{aut}(\Gamma)$,

$$m(y) = m(h_1(y)). \tag{5.4}$$

Adding equations of the form (5.1) for both y and $h_1(y)$ and applying (5.3) and (5.4) we obtain

$$m(y) = 2 | \Gamma | - | \Sigma | - 2 \tag{5.5}$$

and so $\mathscr{G}$ is edge regular.

Now the hypotheses of Theorem 2 are satisfied and so $\mathscr{G}$, and (using Lemma 3.1, iii) even the mapping h_1, are uniquely determined. Thus G^* is unique (see the construction in [7]). The groups listed then account for each case. (Details identifying the G^* so constructed appear in [8].)

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